Correlations in interacting systems with a network topology

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We study pair correlations in interacting systems placed on complex networks. We show that usually in these systems, pair correlations between interacting objects (e.g., spins), separated by a distance ℓ, decay, on average, faster than $1/(\ell z_\ell)$. Here z_ℓ is the mean number of the ℓth nearest neighbors of a vertex in a network. This behavior, in particular, leads to a dramatic weakening of correlations between second and more distant neighbors on networks with fat-tailed degree distributions, which have a divergent number z_2 in the infinite network limit. In large networks of this kind, only pair correlations between the nearest neighbors are actually observable. We find the pair correlation function of the Ising model on a complex network. This exact result is confirmed by a phenomenological approach.

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I. INTRODUCTION

The generic features of real-world networks (the Internet, the WWW, biological, social and economical networks and many others) are a complex organization of their connections and the small-world phenomenon [1–8]. In the networks with the small-world effect, a mean intervertex distance grows with a total number of vertices, N, slower than any power of N, e.g., grows as $\ln N$ [9]. Concerning economical and social networks, the small-world property is often considered as an evidence for the growing interrelations and globalization.

Many of the real-world networks are formed from interacting objects and demonstrate complicated dynamics. The dependence of correlations between interacting objects on time and distance provides a useful information about the network dynamics. In the present paper we discuss general properties of correlations between a pair of interacting objects on a complex network. Recall that in interacting systems on lattices, with a few exceptions, pair correlations decrease exponentially with distance apart from a critical point, where the decrease is power law. Naively, one might expect that the small-world property of a network would enhance pair correlations between distant objects in comparison to lattices. However, it is not the case. We demonstrate that correlations between ℓth nearest neighbors, on average, decay with ℓ as $1/(\ell z_\ell)$ or faster. In networks where the mean intervertex separation $\bar{\ell}(N) \sim \ln N$, this corresponds to the exponential decay of correlations with ℓ (even at the critical point). However, in networks with a divergent mean number of the second nearest neighbors [where $\bar{\ell}(N)$ grows slower than $\ln N$], we observe a dramatic weakening of pair correlations between the second and more distant nearest neighbors. In these networks, only the nearest neighbors are strongly correlated, while the correlations between more distant vertices are suppressed and approach zero in the infinite network limit.

In Sec. II we consider pair correlations in an interacting system defined on the top of a complex network in the framework of a phenomenological approach. This approach allows us to understand general properties of pair correlations irrespective of details of an interacting system and the nature of interactions. In Sec. III we support these results by calculations of the static pair correlation function of the Ising model on a complex network (more precisely, the configuration model of a network [10]).

II. PHENOMENOLOGICAL APPROACH

Let a quantity $X_i(t)$ describe a dynamic process on a network, where the index i labels vertices, and t is time. $x_i(t) \equiv X_i(t) - \langle X_i \rangle$ describes fluctuations around the average value $\langle X_i \rangle$. In an equilibrium state, pair correlations between two arbitrary vertices i and j may be characterized by the following correlation function:

$$G_{ij}(t_1,t_2) = t_0^{-1}\int_0^{t_0} x_i(t+t_1)x_j(t+t_2)dt \equiv \langle x_i(t_1)x_j(t_2)\rangle. \qquad (1)$$

In the present section the brackets $\langle \cdots \rangle$ denote an average over the observation time t_0. The latter must be much larger than the maximum relaxation time of the system under consideration. We have $G_{ij}(t_1,t_2)=G_{ij}(t_1-t_2)$ in the limit $t_0 \to \infty$. In the framework of the Hamiltonian dynamics, one can introduce a generalized field $H_i(t)$ conjugated to $X_i(t)$. The function

$$\chi_{ij}(t_1-t_2) = t_0^{-1}\int_0^{t_0} \partial x_i(t+t_1)/\partial H_j(t+t_2)dt \equiv \langle \partial x_i(t_1)/\partial H_j(t_2)\rangle \qquad (2)$$

is a generalized nonlocal susceptibility which characterizes the averaged response of $x_i(t_1)$ at time t_1 on a field applied at

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a vertex j at time t_2. There is a simple relationship between $G_{ij}(t)$ and $\chi_{ij}(t)$,

$$G_{ij}(t) \propto \chi_{ij}(t), \tag{3}$$

where the coefficient of the proportionality is not important for our purpose.

In the general case, the total susceptibility $\chi(t)$ per vertex has a finite value in the limit $N \to \infty$, where N is the total number of vertices in the network,

$$\chi(t_1 - t_2) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} \left\langle \frac{\partial x_i(t_1)}{\partial H_j(t_2)} \right\rangle \bigg|_{H_1=H_2=\cdots=H} = O(1). \tag{4}$$

The susceptibility $\chi(t)$ may diverge only at a critical point of a phase transition. We rewrite Eq. (4) as follows:

$$\chi(t) = \frac{1}{N}\sum_i \left(\chi_{ii}(t) + \sum_{j:\ell_{ij}=1} \chi_{ij}(t) + \sum_{j:\ell_{ij}=2} \chi_{ij}(t) + \cdots \right). \tag{5}$$

Here, ℓ_{ij} denotes a distance between vertices i and j. The first sum in the parentheses is a sum over the nearest neighbors j of a vertex i, i.e., $\ell_{ij}=1$. The second sum is over the second nearest neighbors, i.e., $\ell_{ij}=2$, and so on. We define the average value of $\chi_{ij}(t)$ at $\ell_{ij}=\ell$:

$$\chi(t,\ell) = \frac{\sum_{i,j:\ell_{ij}=\ell} \chi_{ij}(t)}{\sum_{i,j:\ell_{ij}=\ell} 1} = \frac{\sum_{i,j:\ell_{ij}=\ell} \chi_{ij}(t)}{N z_\ell}, \tag{6}$$

where the sums are over all pairs of vertices with the intervertex distance $\ell_{ij}=\ell$. $z_\ell = N^{-1}\Sigma_{i,j:\ell_{ij}=\ell} 1$ is the mean number of ℓth nearest neighbors. Consequently,

$$\chi(t) = \sum_\ell z_\ell \chi(t,\ell), \tag{7}$$

where $z_0=1$ and $\chi(t,0)$ is the average local susceptibility. In the case of a ferromagnetic interaction the susceptibility $\chi(t,\ell)$ is positive, $\chi(t,\ell) \geqslant 0$. The condition of convergence of the sum in Eq. (7) leads to the following restriction on the magnitude of $\chi(t,\ell)$:

$$\chi(t,\ell) < O\left(\frac{1}{\ell z_\ell}\right). \tag{8}$$

It is important that in the general case, z_ℓ is a function of the network size N. So, Eq. (8) shows how the nonlocal susceptibility (and spacial correlations) varies with N.

In systems where signs of interactions vary at random, the nonlocal susceptibilities $\chi_{ij}(t)$ have random signs. In this case Eq. (8) is not valid. Averaging over all pairs of vertices i,j with a given $\ell_{ij}=\ell$, we arrive at

$$[\overline{\chi_{ij}^2(t,\ell_{ij}=\ell)}]^{1/2} < O(z_\ell^{-1/2}). \tag{9}$$

In accordance to Eq. (3), the ℓ dependence of the average correlation function $G(t,\ell)$ is the same as for the average susceptibility $\chi(t,\ell)$, i.e., it is described by Eqs. (8) or (9).

In general, in a network with a finite second moment $\Sigma_k k^2 P(k)$ of the degree distribution $P(k)$, the mean intervertex distance $\overline{\ell}(N)$ increases with N as $\ln N$. In turn, the mean numbers z_ℓ of ℓth nearest neighbors grow exponentially with ℓ. In particular, in an uncorrelated random network (without degree-degree correlations between nearest neighbor vertices), we have $z_\ell = z_1(z_2/z_1)^{\ell-1}$ where $z_2 = \Sigma k(k-1)P(k)$ [11]. This relationship is also valid for a regular Bethe lattice and a Cayley tree. In accordance with Eq. (8), the nonlocal susceptibility and pair correlations decrease exponentially with ℓ: $\chi(t,\ell), G(t,\ell) \propto \exp(-(\ell-1)/\xi)$. In this case the correlation length is

$$\xi \sim \frac{1}{\ln(z_2/z_1)} \cong \frac{\overline{\ell}(N)}{\ln N}. \tag{10}$$

This equation demonstrates that ξ is determined (or more precisely, restricted from above) by the structure of a network.

Essentially different situation takes place when the degree distribution is fat-tailed, and its second and higher moments diverge at $N \to \infty$, so that $z_{\ell>1}(N \to \infty) \to \infty$. Behavior of the functions $z_\ell(N)$ at large N depends on a specific network model [12,13]. In networks of this kind the mean intervertex distance $\overline{\ell}(N)$ increases slower than $\ln N$. In particular, in scale-free networks, this corresponds to the degree distribution $P(k) \sim k^{-\gamma}$ with exponent $\gamma \leqslant 3$. Consequently, according to Eq. (8) or Eq. (9), pair correlations between the second and more distant nearest neighbors vanish in the limit $N \to \infty$. Only pair correlations between the nearest neighbors, $\ell=1$, are observable in this limit.

When the second moment of the degree distribution diverges, the relationship $z_\ell = z_1(z_2/z_1)^{\ell-1}$ is valid only if the cutoff of the degree distribution increases with N sufficiently slowly. Otherwise, z_ℓ grows with ℓ nonexponentially. In this situation, at large but finite N, the decay of $G(t,\ell)$ with ℓ is also nonexponential. And as a result, the notion of "correlation length" cannot be applied.

We stress that the conclusions of this section are valid for any random network, including networks with high clustering and various correlations.

III. CORRELATIONS IN THE ISING MODEL

Recent investigations have revealed that the critical behavior of the Ising and Potts models on complex networks strongly differs from the standard mean-field behavior on a regular lattice [14–20]. Let us analyze pair correlations in the Ising model on an uncorrelated random complex network.

A. The model

We consider the ferromagnetic Ising model,

$$\mathcal{H} = -J\sum_{\langle ij \rangle} S_i S_j - \sum_i H_i S_i, \tag{11}$$

where $S_i = \pm 1$, and H_i is a local magnetic field at a vertex i on an uncorrelated random complex network. The sum is over edges. The static pair-correlation function

$$G_{ij} \equiv \langle S_i S_j \rangle - \langle S_i \rangle \langle S_j \rangle \tag{12}$$

is related to the nonlocal magnetic susceptibility χ_{ij},

$$\chi_{ij} = \partial M_i / \partial h_j = \beta G_{ij}, \tag{13}$$

where $\beta = 1/T$, T is temperature, $M_i \equiv \langle S_i \rangle$. In this section $\langle \cdots \rangle$ means the statistical average with the Hamiltonian $\mathcal{H}$.

As a substrate, we use the standard model of an uncorrelated random network—the configuration model [10]. This is the maximally random graph with a given degree distribution or, as it is called in graph theory, a labelled random graph with a given degree sequence. It is important that uncorrelated networks have a locally treelike structure. In the large size limit $N \to \infty$, the probability to find a loop of a finite length, which passes through a given vertex, tends to zero. This is valid for a random uncorrelated network with a finite second moment of degree distribution. Furthermore, when the second moment diverges, loops may be neglected in the large size limit if the cutoff of degree distribution increases sufficiently slowly with increasing N (for more detail, see Ref. [6] and the recent paper [21]). In the configuration model, only loops of length much greater than $\overline{\ell}(N)$ are numerous. Equation (10) shows that the correlation length is much smaller than $\overline{\ell}(N)$. This allows us to neglect loops.

An uncorrelated random network may be considered as a random Bethe lattice, which, by definition, has no boundary. In contrast to a Bethe lattice, a Cayley tree contains boundary vertices which are dead ends [22].

B. How to solve the Ising model on a treelike graph

In order to solve a interacting model on a treelike network we use an effective approach which was applied to the Ising and Potts models on regular [22,23] and random [15,24] Bethe lattices.

Consider an arbitrary treelike graph. Consider a spin S_i with k_i adjacent spins S_j, $j = 1, 2, \ldots, k_i$. The vertex i may be treated as a root of a tree. In turn, a nearest neighboring vertex j may be treated as a root of a subtree which grows from the vertex i. In order to characterize this subtree we introduce a quantity

$$g_{ij}(S_i) = \sum_{\{S_n\}=\pm 1} \exp\left(\beta J \sum_{\langle nm \rangle} S_n S_m + \beta J S_i S_j + \beta \sum_n H_n S_n \right). \tag{14}$$

The indices n and m run over spins that belong to the subtree including the spin S_j.

We introduce k_i parameters x_{ij}, $j = 1, 2, \ldots, k_i$, for each vertex i,

$$x_{ij} \equiv g_{ij}(-1)/g_{ij}(+1). \tag{15}$$

Each edge of the graph under consideration is characterized by two parameters, x_{ij} and x_{ji}. In a general case, we have $x_{ij} \neq x_{ji}$ because x_{ij} and x_{ji} characterize different subtrees. In sum, the Ising model on an arbitrary treelike graph is described by $2L = \Sigma_i k_i$ parameters x_{ij}. L is the total number of edges of the graph.

Using Eqs. (14) and (15), the parameter x_{ij} may be related to parameters x_{jl}, $l = 1, 2, \ldots, k_j$, which characterize edges outgoing from the nearest neighboring vertex j [22],

$$x_{ij} = y\left(H_j, \prod_{l=1}^{k_j - 1} x_{jl} \right). \tag{16}$$

The index l numerates the first nearest neighbors of the vertex j, which in turn are second neighbors of the vertex i, i.e., $\ell_{jl} = 1$ and $\ell_{il} = 2$. The function $y(H, x)$ in Eq. (16) is given by the equation

$$y(H, x) = \frac{e^{(-J+H)\beta} + e^{(J-H)\beta} x}{e^{(J+H)\beta} + e^{(-J-H)\beta} x}. \tag{17}$$

In order to find x_{ij}, it is necessary to solve self-consistently a set of $2L$ equations (16). For an arbitrary treelike graph, these equations may be solved numerically, e.g., by use of the population dynamic algorithm, see Ref. [20] where this method has been applied to the Potts model on a treelike graph.

Observable thermodynamic quantities of the Ising model may be written as functions of the parameters x_{ij}. For example, a magnetic moment M_i is given by the following equation:

$$M_i = \left(e^{2\beta H_i} - \prod_{j=1}^{k_i} x_{ij} \right) \Big/ \left(e^{2\beta H_i} + \prod_{j=1}^{k_i} x_{ij} \right). \tag{18}$$

Finally, observables should be averaged over the ensemble of uncorrelated random graphs with a given degree distribution function $P(k)$.

C. Derivation of the pair correlation function

Let us find a nonlocal susceptibility χ_{ij}, Eq. (13), for the distance $\ell_{ij} = \ell$ from i to j. On a treelike graph there is the only shortest path which connects vertices i and j. It starts from i, then goes through vertices $i_1, i_2, \ldots, i_{\ell-1}$ and ends at j. Using Eqs. (18) and (16) we get an exact equation,

$$\chi_{ij} = \frac{\partial M_i}{\partial x_{ii_1}} \frac{\partial x_{ii_1}}{\partial x_{i_1 i_2}} \frac{\partial x_{i_1 i_2}}{\partial x_{i_2 i_3}} \cdots \frac{\partial x_{i_{\ell-2} i_{\ell-1}}}{\partial x_{i_{\ell-1} j}} \frac{\partial x_{i_{\ell-1} j}}{\partial H_j}. \tag{19}$$

At first, for the purpose of comparison, we find a nonlocal susceptibility χ_{ij} of a regular Bethe lattice with a coordination number k.

In a uniform magnetic field $H_1 = H_2 = \cdots = H$ all vertices and all edges in a regular Bethe lattice are equivalent. Therefore, the parameters x_{ij} are equal, i.e., $x_{ij} = x$. Equation (16) takes the form

$$x = y(H, x^{k-1}). \tag{20}$$

This equation determines x as a function of T and H. From Eqs. (18)–(20) we get

$$\chi(\ell) = \frac{1}{k} \frac{\partial M}{\partial x} \left(\frac{1}{(k-1)} \frac{\partial y(H, x^{k-1})}{\partial x} \right)^{\ell-1} \frac{\partial y(H, x^{k-1})}{\partial H}, \tag{21}$$

where $M = (e^{2\beta H} - x^k)/(e^{2\beta H} + x^k)$.

At zero magnetic field, $H = 0$, Eq. (21) gives

$$\chi(\ell) = \frac{4\beta x^k}{(1+x^k)^2}\left(\frac{2x^{k-2}\sinh(2J\beta)}{(e^{J\beta}+e^{-J\beta}x^{k-1})^2}\right)^{\ell}. \tag{22}$$

At temperatures T above the critical temperature $T_c = 2J/\ln[k/(k-2)]$ of the ferromagnetic phase transition, Eq. (20) has the only solution $x=1$. For this solution, Eq. (22) gives

$$\chi(\ell) = \beta G(l) = \beta \tanh^{\ell}(J\beta). \tag{23}$$

This result was obtained for a Cayley tree [25] and regular Bethe lattices [23]. Recall that there is no phase transition in a spin model on a Cayley tree due to the effect of boundary spins.

In the ferromagnetic phase at $T<T_c$ or at an arbitrary T but $H\neq 0$, Eq. (20) has a nontrivial solution $x<1$.

In accordance with Eqs. (22) and (23) the nonlocal susceptibility $\chi(\ell)$ of a regular Bethe lattice decays exponentially with ℓ, $\chi(\ell)\propto \exp(-\ell/\xi)$. The correlation length ξ depends on T and H but does not depend on N. Moreover, ξ has a finite value at all T, including $T=T_c$.

Now let us consider an uncorrelated random network. The Ising model undergoes a ferromagnetic phase transition at a critical temperature $T_c = 2J/\ln[\langle k^2\rangle/(\langle k^2\rangle - 2\langle k\rangle)]$ [15,16]. In the paramagnetic phase at $T\geqslant T_c$ and $H=0$, Eq. (16) has only the trivial solution $x_{ij}=1$. From Eq. (19) we find that the nonlocal susceptibility $\chi_{ij}(\ell_{ij}=\ell)$ has the same temperature dependence, Eq. (23), as that for a regular Bethe lattice or a Cayley tree.

We average the susceptibility $\chi_{ij}(\ell_{ij}=\ell)$, Eq. (19), over the ensemble of uncorrelated random graphs with a given degree distribution function $P(k)$,

$$\chi(\ell) \equiv \overline{\chi_{ij}(\ell_{ij}=\ell)} = \sum_{k_i}\sum_{k_1}\cdots\sum_{k_{l-1}}\sum_{k_j}\chi_{ij} \times \left[\frac{P(k_j)k_j}{z_1}\left(\prod_{n=1}^{\ell-1}\frac{P(k_n)k_n(k_n-1)}{z_2}\right)\frac{P(k_i)k_i}{z_1}\right]. \tag{24}$$

The quantity in the square brackets is the probability that a vertex i of degree k_i is connected with a vertex j of degree k_j by a path that goes through vertices of degrees $k_1, k_2, \ldots, k_{\ell-1}$. Either at $H\neq 0$ or $T<T_c$, an approximate expression for $\chi(\ell)$ may be obtained in the framework of the following approach proposed in Ref. [15]. We introduce positive random parameters h_{ij} instead of the parameters x_{ij}, $x_{ij}=\exp(-h_{ij})$. Assuming that the system is sufficiently close to the critical point, we use the following ansatz [15,24]:

$$\sum_{j=1}^{k} h_{ij} \approx kh + O(k^{1/2}), \tag{25}$$

where $h\equiv \Sigma_i\Sigma_j h_{ij}/Nz_1$ is the mean value of the parameter h_{ij} on the network. Applying the ansatz (25) to Eq. (16), we get a self-consistent equation for h,

$$h = -\frac{1}{z_1}\sum_k P(k)k\ln y(H, e^{-(k-1)h}). \tag{26}$$

The parameter h plays the role of the order parameter. We get $h=0$ in the paramagnetic phase, $T>T_c$, at zero magnetic field $H=0$, while $h\neq 0$ at $H\neq 0$ or $T<T_c$. Note that this ansatz gives an exact description of the critical behavior of the Ising model [15,16]. Far from T_c, one must take into account the fluctuations of h_{ij}, which, however, can be made only numerically as in Ref. [20]. With the ansatz (25), we get

$$\chi(\ell) = \left(\frac{z_1}{z_2}\right)^{(\ell-1)} AB^{\ell-1}C, \tag{27}$$

where

$$A = -\frac{1}{z_1}\sum_k kP(k)\frac{2e^{-(k-1)h}}{(e^{H\beta}+e^{-H\beta-kh})^2}, \tag{28}$$

$$B = \sum_k \frac{k(k-1)P(k)}{z_1}\frac{2e^{-(k-2)h}\sinh(2J\beta)}{(e^{(J+H)\beta}+e^{-(J+H)\beta-(k-1)h})^2}, \tag{29}$$

$$C = -\frac{1}{z_1}\sum_k kP(k)\frac{4\beta e^{-(k-1)h}\sinh(2J\beta)}{(e^{(J+H)\beta}+e^{-(J+H)\beta-(k-1)h})^2}. \tag{30}$$

The quantities A, B, and C are functions of T and H. They have a finite value in the limit $N\to\infty$ even for a network with divergent second moment z_2. The convergence of the sums over k in these equations is ensured by the exponential multiplier e^{-kh}. This conclusion does not depend on the fact that we used the approximation (25). Indeed, substituting Eq. (19) into Eq. (24), one can prove that all the sums over degrees k_n converge both in the ordered state and at $H\neq 0$.

For the regular Bethe lattice we have $P(k)=\delta_{k,K}$. In this case Eqs. (27)–(30) are reduced to Eqs. (21)–(23).

Notice that Eqs. (27)–(30) were derived for an uncorrelated random network with a locally treelike structure. Only in this case the relationship $z_\ell = z_1(z_2/z_1)^{\ell-1}$ is valid. A network with divergent $z_2(N)$ also may have a locally treelike structure if the cutoff of the degree increases with N sufficiently slowly. This depends on a specific network model (see also a discussion at the end of Sec. II).

Equation (27) shows that $\chi(\ell)\propto\exp[-(\ell-1)/\xi]$ with $\xi = [z_2/(z_1 B)]$ in agreement with the result obtained in the preceding section [compare Eqs. (27) and (8)]. If the second moment z_2 of the degree distribution has a finite value in the limit $N\to\infty$, then the correlation length ξ is also finite as well as for the regular Bethe lattice. If z_2 diverges in the infinite size limit, then $\xi\to 0$. Therefore, long-range pair correlations vanish, $G(\ell\geqslant 2)\propto\chi(\ell\geqslant 2)\to 0$.

IV. DISCUSSION AND CONCLUSIONS

Our analysis may be generalized to correlated networks. Many natural networks exhibit correlations between degrees of adjacent vertices, see, for example, Ref. [26]. Note that in large uncorrelated networks, the divergence of the mean number z_ℓ of the ℓth nearest neighbor can occur only simultaneously at all $\ell \geqslant 2$. In contrast, in large correlated networks, it is possible in principle that, say, z_2, z_3 is finite and only $z_{\ell \geqslant 4}$ diverges. In this case, pair correlations are observable between the first, second, and third nearest neighbors and vanish starting from the fourth nearest neighbors in the infinite size limit $N \to \infty$.

It is important to note that the conclusions obtained in Sec. II in the framework of phenomenological approach are quite general and valid for networks with high clustering and various structural correlations. Moreover, these conclusions are valid for a wide range of interactions.

We discussed only the pair correlations between interacting objects (e.g., spins) separated by an intervertex distance ℓ. One should emphasize that the pair correlations in interacting systems on small worlds, unlike lattices, never show critical behavior. Indeed, even at the critical point of an interacting system on a network with the small-world phenomenon, we observe an exponential decay of the pair correlations. Nonetheless, other characteristics of correlations in interacting systems on networks may demonstrate a critical feature. For example, consider the following quantity. The distribution of the full response of a system to a small local field is $P(\varepsilon)=\Sigma_i \delta(\Sigma_j \chi_{ij} - \varepsilon)$, where $\delta(\varepsilon)$ is the delta function. This distribution is a rapidly decreasing function both below and above a phase transition on a network. On the other hand, at the critical point, $P(\varepsilon)$ is a power law [11,27].

One can conclude that a network structure of an interacting system strongly influences the dependence of pair correlations on distance between interacting objects. We have demonstrated that the compactness of a network substrate dramatically suppresses the pair correlations and essentially determines their rapid decay.

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